Position-dependent mass models and their nonlinear characterization

2007 J. Phys. A: Math. Theor. 40 F1041

(http://iopscience.iop.org/1751-8121/40/49/F01)

IOP FTC

IOP Publishing | Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) F1041–F1045 doi:10.1088/1751-8113/40/49/F01

FAST TRACK COMMUNICATION

Position-dependent mass models and their nonlinear characterization

B Bagchi

Department of Applied Mathematics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Kolkata - 700 009, India

E-mail: bbagchi123@rediffmail.com

Received 7 August 2007, in final form 2 November 2007
Published 21 November 2007
Online at stacks.iop.org/JPhysA/40/F1041

Abstract

We consider the specific models of Zhu–Kroemer and BenDaniel–Duke in a sech^2-mass background and point out interesting correspondences with the stationary 1-soliton and 2-soliton solutions of the KdV equation in a supersymmetric framework.

PACS numbers: 03.65.−w, 11.30.Lm, 03.65.Ge, 11.30.Pb

In dealing with position dependent mass (PDM) models controlled by a sech^2-mass profile, we demonstrated [1] recently that, in the framework of a first-order intertwining relationship, such a mass environment generates an infinite sequence of bound states for the conventional free-particle problem. Noting that the intertwining relationships are naturally embedded in the formalism [2] of the so-called supersymmetric quantum mechanics (SUSYQM), we feel tempted to dig a little deeper by choosing to examine the connections between the discrete eigenvalues of such a PDM quantum Hamiltonian (transformed appropriately so that a SUSY structure is evident) and the stationary soliton solutions of the Korteweg–de Vries (KdV) equation that match with the mass function up to a constant of proportionality.

Let us begin with the standard time-independent representation of the PDM Schrödinger equation [3]

$$\left[-\frac{d^2}{dx^2}+\frac{3}{4}\frac{M'^2}{M^2}-\frac{1}{2}\frac{M''}{M}+M(V_{\text{eff}}-\epsilon)\right]\psi=0, \tag{1}$$

where $M(x)$ is the dimensionless equivalence of the mass function $m(x)$ defined by $m(x)=m_0M(x)$, and we have chosen units such that $\hbar=2m_0=1$. The effective potential V_{eff} contains, apart from the given $V(x)$, the real ambiguity parameters α and β whose occurrences are typical in PDM settings:

$$V_{\text{eff}}=V(x)+\frac{1}{2}(\beta+1)\frac{M''}{M^2}-\{\alpha(\alpha+\beta+1)+\beta+1\}\frac{M'^2}{M^3}. \tag{2}$$

Suitable physical choices of α and β have been reported in the literature [4–25], but of particular interest to us are the schemes of Zhu–Kroemer (ZK) [4] [$\alpha = -1/2, \beta = 0$] and BenDaniel–Duke (BDD) [5] [$\alpha = 0, \beta = -1$] which were shown [1] to be dual of each other for the free-particle case $V(x) = V_0$ that is independent of any choice of $M(x)$.

Substituting (2) into (1) and assuming for V_0 the form

$$V_0 = \epsilon - \lambda(\lambda+1)q^2, \qquad \lambda, q \in \mathbf{R}, \tag{3}$$

we can recast (1) to the standard constant-mass Schrödinger equation, namely

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u\right)\psi = 0 \tag{4}$$

with the energy level term missing. In (4), u is given by

$$u = \left[\frac{3}{4} - \{\alpha(\alpha+\beta+1)+\beta+1\}\right]\frac{M'^2}{M^2} + \frac{1}{2}\beta\frac{M''}{M} - \lambda(\lambda+1)Mq^2. \tag{5}$$

However, equation (4) can also be regarded as the linearized partner of the Riccati equation

$$u = v^2 + v' \tag{6}$$

upon putting $v = \frac{\psi'}{\psi}$. The latter is the Cole–Hopf transformation.

A nonlinear connection such as the one given by (6), also known as the Miura map, has an interesting implication. It transfers a solution of the modified KdV equation

$$v_t = 6v^2v' - v''' \tag{7}$$

into a solution of the KdV equation

$$u_t = 6uu' - u''' \tag{8}$$

which is straightforward to check.

The KdV equation has a very rich internal structure [26, 27]. In particular, it admits of a Lax representation $L_t = [B, L]$, where $L = -\partial^2 + u$ is a Schrödinger-like operator and B is given by $B = -4\partial_x^3 + 6u\partial_x + 3u'$. One can solve for L in the from $L(t) = S(t)L(0)S^{-1}(t)$ with $S_t = BS$. The related eigenvalue problem then implies that the spectrum of L is conserved and yields for the KdV an infinite chain of conserved charges.

Noting that the KdV is invariant under the set of transformations

$$t \to t', \qquad x \to x' - 6ct', \qquad u \to u' + c, \tag{9}$$

where c is a constant, the energy levels μ_n can be introduced in (4),

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u\right)\psi = \mu_n\psi. \tag{10}$$

The manner of interplay between the PDM form (5) of u for specific choices of the parameters α, β and the initial condition $u(x, 0) = u_0$ used as inputs to solve for the KdV (as is normally done in the inverse scattering problem) is our point of enquiry.

It can be proved that the discrete eigenvalues μ_n are time independent. For this we have to express the KdV in the conserved form

$$u_t + (-3u^2 + u_{xx})_x = 0 \tag{11}$$

and substitute u from (10) into it. We obtain

$$(\mu_n)_t\psi^2 + (\psi\phi_x - \psi_x\phi)_x = 0, \tag{12}$$

where $\phi = \psi_t + \psi_{xxx} - 3(u+\mu)\psi_x$. On integrating (12) we find $(\mu_n)_t = 0$ where we have employed normalized ψ and considered vanishing asymptotic conditions for ψ and its derivatives. The eigenvalues μ_n are determined using for the potential the initial value u_0 that corresponds to a stationary soliton solution of the KdV equation.

In the context of (10), the Riccati equation (6) is transformed to

$$u = v^2 + v' + \mu, \tag{13}$$

where v as a solution of the generalized MKdV equation

$$v_t = 6(v^2 + \mu)v' - v''' \tag{14}$$

ensures that u evolves according to the KdV equation.

For the 1-soliton and the 2-soliton solutions of the KdV, the corresponding starting solutions u_0 along with ψ, v and the eigenvalues μ_n $(n = 1, 2)$ are given by

$$\begin{aligned}
&\text{1-soliton: } u_0^{(1)} = -2q^2 \operatorname{sech}^2 qx,\\
&\qquad \psi_1 = \tfrac{1}{\sqrt{2}} \operatorname{sech} qx, \qquad v^{(1)} = -q \tanh qx, \qquad \mu_1 = -q^2\\
&\text{2-soliton: } u_0^{(2)} = -6q^2 \operatorname{sech}^2 qx,\\
&\qquad \psi_2^{(a)} = \tfrac{\sqrt{3}}{2} \operatorname{sech}^2 qx, \qquad v_2^{(a)} = -2q \tanh qx, \qquad \mu_2^{(a)} = -4q^2
\end{aligned} \tag{15}$$

and

$$\psi_2^{(b)} = \frac{\sqrt{3}}{2} \operatorname{sech} qx \tanh qx, \qquad v_2^{(b)} = q\frac{1 - 2\tanh^2 qx}{\tanh qx}, \qquad \mu_2^{(b)} = -q^2,$$

where note that for one discrete value of the Schrödinger equation (10), there exists a 1-soliton solution and vice versa. Similarly for the 2-soliton case. Here the ψ's are normalized.

The results in (15), which can also be extended to the N-soliton case, have been obtained by solving the eigenvalue problem for the Schrödinger equation (10). The solutions $u_0^{(1)}$ and $u_0^{(2)}$ act in (10) as the reflectionless potentials. The inverse scattering method, which exploits this reflectionless feature, determines the evolution of the scattering parameters. Subsequently, the Gel͗fand–Levitan integral equation is solved to obtain the solution $u(x, t)$ of the KdV equation.

Turning now to the PDM induced u given by (5), we immediately recognize from (10) that for the choice of the mass function $M(x) = \operatorname{sech}^2$ qx, the ZK scheme yields the 1-soliton results $u_0^{(1)}$, $\psi_1(\mu = -q^2)$ corresponding to $\lambda = 1, -2$ and the 2-soliton result $u_0^{(2)}$, $\psi_2^{(a)}(\mu = -4q^2)$ and $\psi_2^{(b)}(\mu = -q^2)$ corresponding to $\lambda = 2, -3$.

On the other hand, the BDD scheme is consistent with the form

$$u = q^2(1 - 2\operatorname{sech}^2 qx), \qquad \lambda = 1, -2 \tag{16}$$

for $\psi_1(\mu = 0)$ and

$$u = q^2(1 - 6\operatorname{sech}^2 qx), \qquad \lambda = 2, -3 \tag{17}$$

for both the sets $\psi_2^{(a)}(\mu = -3q^2)$ and $\psi_2^{(b)}(\mu = 0)$.

To interpret the above results, a few remarks on SUSY are in order [28]. We first of all verify that not only (13) but also $u = v^2 - v' + \mu$ carries a solution of the generalized KdV (14) into a solution of the KdV.

Denoting

$$V^{(\pm)} \equiv u^{\pm} - \mu = v^2 \mp v', \tag{18}$$

we note that the combination $V^{(\pm)}$ can be identified as the usual partner potentials of SUSYQM.

To examine the role of $V^{(\pm)}$ in the present context, let there be a Hamiltonian H_1 with potential V_1 that is asymptotically vanishing and having a set of n discrete eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$. If we define $V^+ = V_1 - \mu_{n+1}$ then, in unbroken SUSY, we at once know that the spectra of $V^{(+)}$ and those of $V^{(-)}$ are one to one except that the latter has an additional

$\mu = 0$ state. In other words, the eigenvalues of $V^{(-)}$ are $\mu_1 - \mu_{n+1}, \mu_2 - \mu_{n+1}, \ldots, \mu_n - \mu_{n+1}$ and 0. This means that the Hamiltonian H_2 with potential V_2 defined by $V_2 = V^{(-)} + \mu_{n+1}$ has $(n+1)$ discrete eigenvalues $\mu_1, \mu_2, \ldots, \mu_n, \mu_{n+1}$.

Let us apply the above ideas to the simple case of $V_1 = 0$ and generate the corresponding potential V_2 with a single bound state with $\mu = -q^2$ [29]. We have $V^{(+)} = v^2 - v' = q^2 > 0$: in other words, $V^{(+)}$ has no bound state at all. Solving we get $v = -q \tanh qx$ (i.e. the 1-soliton result) which in turn gives $V^{(-)} = q^2(1 - 2\,\mathrm{sech}^2\, qx)$ that supports a zero energy ($\mu = 0$) bound state $\psi_0 \sim \mathrm{sech}\, qx$:

$$H_- \psi_0 = \psi_0'' + (v^2 + v')\psi_0. \tag{19}$$

Thus $V_2 = -2q^2\,\mathrm{sech}^2\, qx$ has a single bound state.

We immediately recognize V_2 and $V^{(-)}$ to be the PDM potential u for the ZK and BDD schemes respectively corresponding to the 1-soliton case. The same is true for the 2-soliton results with v matching with the 2-soliton solutions and $V^{(-)}$ emerging similar to (17).

One-dimensional supersymmetric approach to PDM quantum systems has been explored before in PDM scenarios. The partner potentials were found to obey [22] the same PDM dependence but in different potentials. The approach of this work is however different in spirit from such a viewpoint in that we have sought to establish a link between a hierarchy of reflectionless potentials (corresponding to the stationary soliton solutions of the KdV) with an arbitrary bound-state spectrum and those of SUSY in PDM models for suitable values of the ambiguity parameters. Our starting potential pertaining to the free-particle case $V(x) = V_0$ can be made to coincide with $V^{(+)}$ by choosing, for example, $\epsilon = 3q^2$ in the 1-soliton case and $\epsilon = 12q^2$ in the 2-soliton case.

Finally, we can extend our treatment to other special cases of the effective potential V_{eff}, namely those of the Bastard [30] and Li and Kuhn (redistributed) [31] Hamiltonians. For the 1-soliton result of (15), u for the Bastard scheme is $u = -q^2(1+3\,\mathrm{sech}^2\, qx)(\mu = -2q^2)$ while for the 2-soliton results given by (16), u turns out to be $-q^2(1+6\,\mathrm{sech}^2\, qx)$ both for $\psi_2^{(a)}$ and $\psi_2^{(b)}$, with an associated μ-value of $\mu = -5q^2$ and $\mu = -2q^2$, respectively. However, in the Bastard model λ is non-integral. A non-integral λ also emerges in the Li–Kuhn scheme where we find $u = -2q^2\,\mathrm{sech}^2\, qx(\mu = -q^2)$ for the 1-soliton solution and $u = -6q^2\,\mathrm{sech}^2\, qx$ for both the 2-soliton solutions of $\psi_2^{(a)}(\mu = -4q^2)$ and $\psi_2^{(b)}(\mu = -q^2)$.

Acknowledgment

I thank PS Gorain for discussions.

References

[1] Bagchi B, Gorain P, Quesne C and Roychoudhury R 2004 *Mod. Phys. Lett.* A **19** 2765
[2] See, for example, Bagchi B K 2000 *Supersymmetry in Quantum and Classical Mechanics* (Florida: Chapman and Hall)
[3] von Roos O 1983 *Phys. Rev.* B **27** 7547
[4] Zhu Q-G and Kroemer H 1983 *Phys. Rev.* B **27** 3519
[5] BenDaniel D J and Duke C B 1966 *Phys. Rev.* B **152** 683
[6] Arias de Saavedra F, Boronat J, Polls A and Fabrocini A 1994 *Phys. Rev.* B **50** 4248
[7] Puente A, Serra Ll and Casas M 1994 *Z. Phys.* D **31** 283
[8] Lévy-Lebond J-M 1995 *Phys. Rev.* A **52** 1845
[9] Yung K C and Yee J H 1994 *Phys. Rev.* A **50** 104
[10] Chetouani L, Dekar L and Hammann T F 1995 *Phys. Rev.* A **52** 82
[11] Dekar L, Chetouani L and Hammann T F 1998 *J. Math. Phys.* **39** 2551
[12] Dekar L, Chetouani L and Hammann T F 1999 *Phys. Rev.* A **59** 107

[13] Plastino A R, Puente A, Casas M, Garcias F and Plastino A 2000 *Rev. Mex. Fis.* **46** 78
[14] Koç R, Koca M and Körcük E 2002 *J. Phys. A: Math. Gen.* **35** L527
[15] Gönül B, Özer O, Gönül B and Üzgün F 2002 *Mod. Phys. Lett.* A **17** 2453
[16] Alhaidari A D 2002 *Phys. Rev.* A **66** 042116
[17] de Souza Dutra A and Almeida C A S 2000 *Phys. Lett.* A **275** 25
[18] Mustafa O and Mazharimousavi S H 2006 *J. Phys. A: Math. Gen.* **39** 10537
[19] Yu J, Dong S-H and Sun G-H 2004 *Phys. Lett.* A **322** 290
[20] Roy B and Roy P 2002 *J. Phys. A: Math. Gen.* **35** 3961
[21] Milanović V and Ikonić Z 1999 *J. Phys. A: Math. Gen.* **32** 7001
[22] Plastino A R, Rigo A, Casas M, Garcias F and Plastino A 1998 *Phys. Rev.* A **60** 4318
[23] Gönül B, Gönül B, Tuctu D and Özer O 2002 *Mod. Phys. Lett.* A **17** 2057
[24] Cruz y Czuz S, Negro J and Nieto L M 2007 *Phys. Lett.* A **369** 400
[25] Pena J J *et al* 2007 *Int. J. Quantum Chem.* **107** 3039
[26] Miura R M 1976 *SIAM Rev.* **18** 412
[27] Bhatnagar P L 1979 *Nonlinear Waves in One-dimensional Dispersive Systems* (Delhi: Oxford University Press)
[28] Bagchi B 1990 *Int. J. Mod. Phys.* A **5** 1763
[29] Grant A K and Rosner J L 1994 *J. Math. Phys* **35** 1
[30] Bastard G 1981 *Phys. Rev.* B **24** 5693
[31] Li T L and Kuhn K J 1993 *Phys. Rev.* B **47** 12760